Announcements

1) HW # 5 up on CTools, 15 point problems coming later today

2) Chapters from Spivak's "Calculus on Manifolds" up under "Resources".
Read the part on forms.

Implicit Function Theorem

Let $f: E \subseteq \mathbb{R}^{n+m}$

into $\mathbb{R}^n$, differentiable

and f' continuous on E.

Suppose $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$,

$f(a,b) = 0$.

Let $A = f'(a,b)$ and define,

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$(A_x)h = A(h, \underbrace{0}_{\text{in } \mathbb{R}^m})$$

$(h \in \mathbb{R}^n)$

$$(A_y)k = A(\underbrace{0}_{\text{in } \mathbb{R}^n}, k)$$

If A_x is invertible, then $\exists$ open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$, with $(a,b) \in U$ and $b \in W$ such that

$\forall\ y \in W$, there is a unique x with

$$\boxed{(x,y) \in U \text{ and } f(x,y) = 0.}$$

Defining $x = g(y)$, then g is differentiable on W with g' continuous, $g(b) = a$, and $g'(b) = -(A_x)^{-1} A_y$

Example 1: $n = m = 1$.

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = x^2 + y^2 - 1.$$

Suppose $f(a,b) = 0$.

$$a^2 + b^2 - 1 = 0, \text{ so}$$

$$a^2 + b^2 = 1.$$

In one-variable calculus, they say that given $x^2+y^2=1$, we can differentiate "implicitly" and find $\frac{dy}{dx}$:

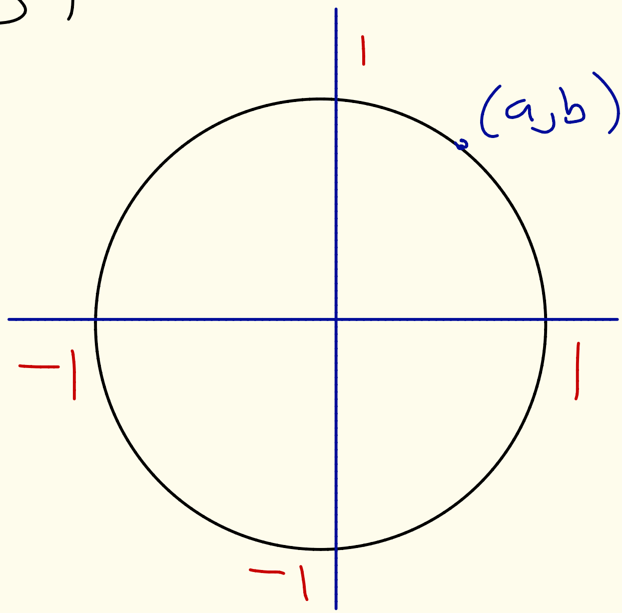
$$2x+2y\frac{dy}{dx}=0,$$

So $\frac{dy}{dx}=-\frac{x}{y}$

except when $y=0$!

In other words, we can do this precisely when: if $a^2+b^2=1$, $b \neq 0$.

Graph



Every neighborhood about $(-1,0)$ or $(1,0)$ contains a portion of the graph that fails the vertical line test and so you can never write $y = g(x)$ for a function $g : \mathbb{R} \to \mathbb{R}$ on such a set.

Note

$$-\frac{x}{y} = \frac{-2x}{2y}$$

$$= \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

If $\frac{\partial f}{\partial y} \neq 0$, then we have invertibility $\Rightarrow$ we can solve for y in terms of x

If $y>0$:

$$y = \sqrt{1-x^2}$$

If $y<0$,

$$y = -\sqrt{1-x^2}$$

but only in a neighborhood

about (a,b). $(b>0, b<0)$

Proposition: (IFT for linear maps)

Let $A: \mathbb{R}^{n+m} \to \mathbb{R}^n$

linear and suppose

A_x is invertible.

Then for every $k \in \mathbb{R}^m$,

there is a unique h in $\mathbb{R}^n$

with $A(h,k) = 0$.

Moreover, $h = -A_x^{-1} A_y k$.

Proof: We may decompose

$$A(h,k)$$

$$= A(h,0) + A(0,k)$$

$$= A_x h + A_y k$$

by definition of A_x, A_y.

If $0 = A(h,k)$

$$= A_x h + A_y k$$

then

$$-A_y k = A_x h$$ and

since A_x is invertible,

we may multiply by A_x^{-1}

to get

$$-A_x^{-1} A_y k = h. \quad \square$$

Proof of Inverse Function Theorem:-

Reduction:- Let $n=m=1$.

$f: E \subseteq \mathbb{R}^2 \to \mathbb{R}$

differentiable, f' continuous on E.

Then

$$f'(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)$$

Then

$$A_x = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$A_y = \frac{\partial f}{\partial y}(x_0, y_0).$$

Step 1: (the trick)

Define

$h: E \to \mathbb{R}^2$ by

$$h(x,y) = (f(x,y), y)$$

The same trick works for the general case.

If we believe h' exists on E, then

$$h'(x_0,y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ 0 & 1 \end{bmatrix}$$

$$\det(h'(x_0,y_0)) = \frac{\partial f}{\partial x}(x_0,y_0)$$

If $(x_0,y_0)=(a,b)$,

then $\frac{\partial f}{\partial x}(a,b) \neq 0$.

By the Inverse Function Theorem, $\exists$ an open ball U containing (a,b) on which h is invertible. Moreover, all conclusions of the inverse function (note h' is continuous on E since f' is) apply.

Let

$$W = \{ y \mid (0, y) \in h(U) \}$$

(we know $h(U)$ is open by the Inverse Function Theorem).

Note $b \in W$ since

$$h(a,b) = (f(a,b), b) = (0, b) \in h(U)$$

since $(a,b) \in U$.

Let's show W is open.

Choose $y_0 \in W$. Then

$\exists$ a $\delta > 0$ with

$$B((0,y_0),\delta) \subseteq h(U)$$

Since $(0,y_0) \in h(U)$ and $h(U)$ is open. Since $B((0,y_0),\delta)$

is convex, we know

$$(0,y_0) + (0,t) \in B((0,y_0),\delta)$$

$\forall$ t, $0 \leq \|t\|_2 < \delta$.

Let

$$W_0 = \{ y \mid \|y - y_0\|_2 < \delta \}.$$

By the calculation from the previous page,

$W_0 \subseteq W \Rightarrow W$ is open.

Then we know

h is invertible

on U, and therefore

if we set

$$h(x_0, y_0) = h(x_1, y_1)$$

for $(x_1, y_1), (x_0, y_0) \in U$,

$$x_0 = x_1, \quad y_0 = y_1.$$

This implies that if

$$f(x_1, y) = f(x_2, y),$$

then, $x_1 = x_2$.

So if $y \in W$, we can define $g(y) = x$ by letting x be this unique point.